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CRITICISMS AND DISCUSSIONS.

A MATHEMATICAL STUDY OF MAGIC SQUARES.

A NEW ANALYSIS.

Magic squares are not simple puzzles to be solved by the old rule of "Try and try again," but are visible results of "order" as applied to numbers. Their construction is therefore governed by laws that are as fixed and immutable as the laws of geometry.

It will be the object of this essay to investigate these laws, and evolve certain rules therefrom. Many rules have already been published by which various magic squares may be constructed, but they do not seem to cover the ground comprehensively. It is the belief of the writer that the rules herein given will be competent to produce all forms of 3×3 and 4×4 squares with their compounds, and

a	b	c
d	e	g
h	m	n

Fig. 1.

x 2v y	x	x v 2y
x 2y	x v y	x 2v
x v	x 2v 2y	x y

Fig. 2.

8	1	6
3	5	7
4	9	2

Fig. 3.

23	2	20
12	15	18
10	28	7

Fig. 4.

also that the principles enunciated will apply largely to all other magic squares.

Let Fig. 1 represent a 3×3 magic square. By inspection we note that:

$$h + c = b + m$$
$$\text{and } h + m = g + c$$
$$\text{therefore } 2h = b + g$$

In this way four equations may be evolved as follows:

$$2h = b + g$$
$$2n = b + d$$

$$2c = d + m$$
$$2a = m + g$$

It will be seen that the first terms of these equations are the quantities which occur in the four corner cells, and therefore that the quantity in each corner cell is a mean between the two quantities in the two opposite cells that are located in the middle of the outside rows. It is therefore evident that the least quantity in the magic square must occupy a middle cell in one of the four outside rows, and that it cannot occupy a corner cell.

Since the middle cell of an outside row must be occupied by the least quantity, and since any of these cells may be made the middle cell of the upper row by rotating the square, we may consider this cell to be so occupied.

Having thus located the least quantity in the square it is plain that the next higher quantity must be placed in one of the lower corner cells, and since a simple reflection in a mirror would reverse the position of the lower corner cells, it follows that the second smallest quantity may occupy *either* of these corner cells. Next we may write more equations as follows:

$$a + e + n = S \text{ (or summation)}$$
$$d + e + g = S$$
$$h + e + c = S$$

also

$$a + d + h = S$$
$$n + g + c = S$$

therefore

$$3e = S$$

and

$$e = S/3$$

Hence the quantity in the central cell is an arithmetical mean between any two quantities with which it forms a straight row or column.

With these facts in view a magic square may now be constructed as shown in Fig. 2.

Let x, representing the least quantity, be placed in the middle upper cell, and $x + y$ in the lower right-hand corner cell, y being the increment over x.

Since $x + y$ is the mean between x and the quantity in the left-hand central cell, this cell must evidently contain $x + 2y$.

Now writing $x + v$ in the lower left-hand corner cell, (considering v as the increment over x) it follows that the central right-hand cell must contain $x + 2v$.

Next, as the quantity in the central cell in the square is a mean between $x + 2y$ and $x + 2v$, it must be filled with $x + v + y$. It now follows that the lower central cell must contain $x + 2v + 2y$, and the upper left-hand corner cell $x + 2v + y$, and finally the upper right-hand corner cell must contain $x + v + 2y$, thus completing the square which necessarily must have magic qualifications with any conceivable values which may be assigned to x, v, and y.

We may now proceed to give values to x, v, and y which will produce a 3×3 magic square containing the numbers 1 to 9 inclusive in arithmetical progression. Evidently x must equal 1, and as there must be a number 2, either v or y must equal 1 also.

Assuming $y = 1$, if $v = 1$ or 2, duplicate numbers would result, therefore v must equal at *least* 3.

In the square under consideration the central number must be 5 and as this number is composed of $x + y + v$, therefore v must equal 3. Using these values, viz., $x = 1$, $y = 1$ and $v = 3$, the familiar 3×3 magic square shown in Fig. 3 is produced.

It is important to recognize the fact that although in Fig. 3 the series of numbers used has an initial number of 1, and also a constant increment of 1, yet this may be considered as only an accidental feature pertaining to this particular square, the real fact being that *a magic square of* 3×3 *is always composed of three sets of three numbers each.* The difference between the numbers of each trio is uniform, but the difference between the last term of one trio and the first term of the next trio is not necessarily the same as the difference between the numbers of the trios.

For example, if $x = 2$, $y = 5$ and $v = 8$, the resulting square will be as shown in Fig. 4.

The trios in this square are as follows:

2 — 7 — 12
10 — 15 — 20
18 — 23 — 28

The difference between the numbers of these trios is $y = 5$, and the difference between the homologous numbers is $v = 8$.

A recognition of these two sets of increments is essential to the proper understanding of the magic square. Their existence is masked in the 3×3 square shown in Fig. 3 by the more or less accidental quality that in this particular square the difference between ad-

x 2s t 2v y	x 2s t	x 2s t v 2y	x 2v y	x	x v 2y	x s 2t 2v y	x s 2t	x s 2t v 2y
x 2s t 2y	x 2s t v y	x 2s t 2v	x 2y	x v y	x 2v	x s 2t 2y	x s 2t y	x s 2t 2v
x 2s t v	x 2s t 2v 2y	x 2s t y	x v	x 2y 2v	x y	x s 2t v	x s 2t 2v 2y	x s 2t y
x 2t 2v y	x 2t	x 2t v 2y	x s t 2v y	x s t	x s t v 2y	x 2s 2v y	x 2s	x 2s v 2y
x 2t 2y	x 2t v y	x 2t 2v	x s t 2y	x s t v y	x s t 2v	x 2s 2y	x 2s v y	x 2s 2v
x 2t v	x 2t 2v 2y	x 2t y	x s t v	x s t 2v 2y	x s t y	x 2s v	x 2s 2v 2y	x 2s y
x s 2v y	x s	x s v 2y	x 2s 2t 2v y	x 2s 2t	x 2s 2t v 2y	x t 2v y	x t	x t v 2y
x s 2y	x s v y	x s 2v	x 2s 2t 2y	x 2s 2t v y	x 2s 2t 2v	x t 2y	x t v y	x t 2v
x s v	x s 2v 2y	x s y	x 2s 2t v	x 2s 2t 2v 2y	x 2s 2t y	x t v	x t 2v 2y	x t y

Fig. 5.

	x 2s t			x			x s 2t	
	x 2t			x s t			x 2s	
	x s			x 2s 2t			x t	

Fig. 6.

jacent numbers is always 1. Nevertheless the square given in Fig. 3 is really made up of three trios, as follows:

1st trio 1 — 2 — 3
2nd " 4 — 5 — 6
3rd " 7 — 8 — 9

in which the difference between the numbers of the trios is $y = 1$, and the difference between the homologous numbers is $v = 3$. Furthermore it is simply an *accidental* quality of this particular square that the difference between the last term of a trio and the first term of the next trio is 1.

Having thus acquired a clear conception of the structure of a 3×3 magic square, we are in a position to examine a 9×9 compound square intelligently, this square being only an expansion of the 3×3 square, and governed by the same constructive rules.

Referring to Fig. 6 the upper middle cells of the nine subsquares may first be filled in the same way that the nine cells in Fig. 2 were filled, using for this purpose the terms, x, t, and s. Using these as the initial terms of the subsquares the square may then be completed, using y as the increment between the terms of each trio, and v as the increment between the homologous terms of the trios. The result is shown in Fig. 5, *in which the assignment of any values to x, y, v, t and s, will yield a perfect, compound* 9×9 *square.*

Values may now be assigned to x, y, v, t and s which will produce the series 1 to 81 inclusive. As stated before in connection with the 3×3 square, x must naturally equal 1, and in order to produce 2, one of the remaining symbols must equal 1. In order to avoid duplicates, the next larger number must at least equal 3, and by the same process the next must not be less than 9 and the remaining one not less than 27. Because $1 + 1 + 3 + 9 + 27 = 41$, which is the middle number of the series 1 — 81, therefore just these values must be assigned to the five symbols used in the construction of the square. The only symbol whose value is fixed, however, is x, the other four symbols may have the values 1 — 3 — 9 or 27 assigned to them indiscriminately, thus producing all the possible variations of a 9×9 compound magic square.

If v is first made 1 and $y = 2$, and afterwards y is made 1 and $v = 2$, the resulting squares will be simply reflections of each other, etc. Six fundamental forms of 9×9 compound magic squares may be constructed as shown in Figs. 7, 8, and 9.

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

71	8	53	64	1	46	69	6	51
26	44	62	19	37	55	24	42	60
35	80	17	28	73	10	33	78	15
66	3	48	68	5	50	70	7	52
21	39	57	23	41	59	25	43	61
30	75	12	32	77	14	34	79	16
67	4	49	72	9	54	65	2	47
22	40	58	27	45	63	20	38	56
31	76	13	36	81	18	29	74	11

Fig. 7.

77	22	51	56	1	30	71	16	45
24	50	76	3	29	55	18	44	70
49	78	23	28	57	2	43	72	17
62	7	36	68	13	42	74	19	48
9	35	61	15	41	67	21	47	73
34	63	8	40	69	14	46	75	20
65	10	39	80	25	54	59	4	33
12	38	64	27	53	79	6	32	58
37	66	11	52	81	26	31	60	5

77	56	71	22	1	16	51	30	45
62	68	74	7	13	19	36	42	48
65	80	59	10	25	4	39	54	33
24	3	18	50	29	44	76	55	70
9	15	21	35	41	47	61	67	73
12	27	6	38	53	32	64	79	58
49	28	43	78	57	72	23	2	17
34	40	46	63	69	75	8	14	20
37	52	31	66	81	60	11	26	5

Fig. 8.

77	58	69	20	1	12	53	34	45
60	68	76	3	11	19	36	44	52
67	78	59	10	21	2	43	54	35
26	7	18	50	31	42	74	55	66
9	17	25	33	41	49	57	65	73
16	27	8	40	51	32	64	75	56
47	28	39	80	61	72	23	4	15
30	38	46	63	71	79	6	14	22
37	48	29	70	81	62	13	24	5

77	20	53	58	1	34	69	12	45
26	50	74	7	31	55	18	42	66
47	80	23	28	61	4	39	72	15
60	3	36	68	11	44	76	19	52
9	33	57	17	41	65	25	49	73
30	63	6	38	71	14	46	79	22
67	10	43	78	21	54	59	2	35
16	40	64	27	51	75	8	32	56
37	70	13	48	81	24	29	62	5

Fig. 9.

It will be noted that these are arranged in three groups of two squares each on account of the curious fact that the squares in each pair are mutually convertible into each other by the following process:

If the homologous cells of each 3×3 subsquare be taken in the order as they occur in the 9×9 square, and a 3×3 square made therefrom, a new magic 3×3 square will result. And if this process is followed with all the cells and the resulting nine 3×3 squares are arranged in magic square order a new 9×9 compound square will result.

For example, referring to the upper square in Fig. 7, if the numbers in the central cells of the nine 3×3 subsquares are arranged in magic square order, the resulting square will be the central 3×3 square in the lower 9×9 square in Fig. 7. This law holds good in each of the three groups of two squares (Figs. 7, 8 and 9) and no fundamental forms other than these can be constructed.

a	b	c	d
g	h	n	m
n	o	p	s
t	v	x	y

Fig. 10.

a x			a v
			b x
			c x
g x	a y	a t	g v

Fig. 11.

a x	g y	g t	a v
b v	c t	c y	b x
c v	b t	b y	c x
g x	a y	a t	g v

Fig. 12,

1	8	12	13
14	11	7	2
15	10	6	3
4	5	9	16

Fig. 13.

The question may be asked: How many variations of 9×9 compound magic squares can be made? Since each subsquare may assume any of eight aspects without disturbing the general order of the complete square, and since there are six radically different, or fundamental forms obtainable, the number of possible variations is 6×8^9!

We may now proceed to analyze the construction of a 4×4 magic square as represented in Fig. 10. From our knowledge of this square and its qualifications we are enabled to write four equations as follows:

$$a + h + p + y = S \text{ (Summation)}$$
$$g + h + n + m = S$$
$$n + o + p + s = S$$
$$t + o + n + d = S$$

By inspection of Fig. 10 it is seen that the sum of the initial terms of these four equations equals S, and likewise that the sum of their final terms also equals S. Hence $h + n + o + p = S$. It therefore follows:

1st. That *the sum of the terms contained in the inside* 2×2 *square of a* 4×4 *square is equal to* S.

2d. Because the middle terms of the two diagonal columns compose this inside 2×2 square, their end terms, or *the terms in the four corner cells of the* 4×4 *square must also equal* S, or:

$$a + d + t + y = S$$

3d. Because the two middle terms of each of the two inside columns (either horizontal or perpendicular) also compose the central 2×2 square, *their four end terms must likewise equal* S.

4th. We may now write the following equations:

$$b + c + v + x = S$$
$$b + c + a + d = S$$

therefore

$$a + d = v + x,$$

which shows that *the sum of the terms in any two contiguous corner cells is equal to the sum of the terms in the two middle cells in the opposite outside column.*

5th. Because

$$g + h + n + m = S$$

and

$$o + h + n + p = S$$

it follows that

$$g + m = o + p$$

or, that *the sum of the two end terms of any inside column,* (either horizontal or perpendicular) *is equal to the sum of the two middle terms in the other parallel column.*

6th. Since

$$t + o + n + d = S$$

and

$$h + o + n + p = S$$

therefore

$$t + d = h + p$$

or *the sum of the two end terms of a diagonal column is equal to the sum of the two inside terms of the other diagonal column.*

These six laws govern *all* 4×4 magic squares whether they are perfect or imperfect, but perfect 4×4 squares also possess the additional feature that the sum of the numbers in any two cells that are equally distant from the center and symmetrically opposite to each other in the square equals $S/2$.

With these rules before us we may now construct a perfect 4×4 magic square. Referring to Fig. 11, in the upper left-hand corner cell we will place a number which may be represented by $a + x$, and in the right-hand upper corner a number represented by $a + v$. Also in the central cells of the lower row we will write numbers represented respectively by $a + y$ and $a + t$. Then in the lower left-hand cell we will place a number represented by $g + x$, and in the central cells of the outer right-hand column numbers represented respectively by $b + x$ and $c + x$, and because the square is to be perfect, we must write in the lower right-hand corner a number represented by $g + v$.

The unfinished perfect 4×4 square thus made may now be studied by the light of the laws previously given.

By inspection we see that

$$a + g = b + c$$

and

$$x + v = y + t$$

We also see that the central cells of the upper row should be occupied by the symbol g together with x and v, by law 4, but if thus occupied, duplicate numbers would result.

It has, however, been just shown that

$$x + v = y + t$$

and therefore g may be combined with y and t, *thus producing diverse numbers, and still remaining correct in summation.*

Seeing that the square is to be perfect, the cell which is symmetrically opposite to that occupied by $a + y$, must be filled by a number which will produce with $a + y$, a number equal to $(a + x) + (g + v)$, which will be $g + t$, because

$$x + v = y + t$$

In the same way the next cell to the left must be filled with $g + y$, and we may similarly fill the two inner cells of the left-hand outer column with $b + v$ and $c + v$.

By like simple calculations all the remaining empty cells may be filled, thus completing the 4×4 square shown in Fig. 12.

We will now proceed to show what numbers may be assigned to the eight symbols used in Fig. 12 to produce a perfect 4×4 magic square containing the numbers 1 to 16.

It is evident that some pair of symbols must equal 1 and therefore that one of the two symbols must equal 1 and the other must equal 0, (minus and fractional quantities being excluded).

It is also evident that because $a + g = b + c$, if a is the *smallest* number in the series, g must be the *largest*, and therefore the four numbers represented by a, b, c, g must form a series in which the means equal the extremes. In like manner x, y, t, v must also form another similar series.

Supposing now that $x = 1$ and $a = 0$, then $g + v$ must equal 16, and since b and c are *each less* than g, and must be also *diverse from each other*, we find that g *cannot be less than* 3. Supposing therefore that $g = 3$, then because $a + g = 3 = b + c$, it is evident that b must equal 1 (or 2) and c must equal 2 (or 1). The four quantities a, b, c, g may therefore be assigned values as given below.

$a = 0$	$x = 1$
$b = 1$	$y = 5$
$c = 2$	$t = 9$
$g = 3$	$v = 13$

As $g + v = 16$, v must equal 13 and $y + t$ must equal 14. By inspection it is seen that either y or t must equal 5, and assigning this number to y, t becomes 9, or vice versa.

With these values assigned to the symbols, Fig. 12 will develop the perfect 4×4 square shown in Fig. 13.

The possible number of diverse 4×4 magic squares which may be constructed using the numbers 1 to 16 inclusive has been variously estimated by different writers, 880 changes having been heretofore considered the maximum number. It can however be easily proven that no less than 4352 of these squares may be constructed, which will be demonstrated under the next heading.

A STUDY OF THE POSSIBLE NUMBER OF VARIATIONS IN MAGIC SQUARES.

It has been shown in connection with the 3×3 magic square that there is only one possible arrangement of nine different numbers, which will constitute a magic square.

The 4×4 and all larger squares may however be constructed in great variety, their number of diverse forms increasing in an immense ratio with every increase in the size of square.

Beginning with the 4×4 square, in order to solve the problem of the possible number of variations that may be constructed with the numbers 1 to 16 inclusive, it will be necessary to consider the relative properties of its component elements, which may be conveniently expressed as follows, although there are several other sets of eight numbers whose combinations will yield similar results.

$a = 1$	$x = 0$
$b = 2$	$y = 4$
$c = 3$	$t = 8$
$g = 4$	$v = 12$

As previously stated, it will be seen that

$$a + g = b + c$$

and $x + v = y + t$.

In consequence of this law we find that a column in a 4×4 magic square may contain each of the eight qualities once (as in the diagonal rows of square shown in Fig. 12). In other cases a pair of elements may be lacking, but be represented by another pair, the latter being repeated in the column, (as shown in the two outer vertical columns of Fig. 12). This ability to duplicate some of the elements in place of others that are omitted leads to an enormous amplification of the number of possible variations.

If all the cells in any column are filled, (or any set of four cells, the summation of which is equal to a column) the remainder of the square may then be completed by the rules previously given. This column may therefore be termed a "basic" row or column.

There are four plans by which a basic row may be filled, thus making four classes of squares which may be called Classes, I, II, III and IV.

For the sake of brevity, the symbols a, b, c and g will be termed the "a" elements, and x, y, t and v the "x" elements.

Class I includes those squares in which the basic row is made up of all of the eight elements used once each.

Class II includes those squares in which one of the elements used in the first cell of the basic row is also used in the second cell.

Class III includes those squares in which an element of the first cell in the basic row is also used in the third cell.

Class IV includes those squares in which elements of the first cell in the basic row are also used in the second and third cells.

Class I may be further divided into three Genera as follows:

Genus A comprises those squares in which neither the outer nor inner pair of cells contain either a mean or an extreme pair of "*a*" or "*x*" elements. Fig. 14 represents a basic row of Class I, Genus A.

Genus B comprises those squares in which both the inner and outer pair of cells of the basic row contain a pair of elements as shown in Fig. 15 in which the outer cells contain a pair of "*a*" elements (a and g) and the inner cells also contain a pair of "*a*" elements (b and c).

Genus C comprises those squares in which both pairs of cells

a y	b x	g t	c v

Fig. 14.

a y	b v	c t	g x

Fig. 15.

a x	b t	c y	g v

Fig. 16.

contain two pairs of elements each, as for example when the two outer cells contain a and g, and x and v, and the two inner cells contain b and c, and t and y, as shown in Fig. 16.

Classes II, III, and IV have but one genus each, and there are consequently in all, six different types. To determine the number of specimens which each genus will yield, we will now expand a basic row into a complete square.

a y	b x	g t	c v
g v	c t	a x	b y
c x	g y	b v	a t
b t	a v	c y	g x

Fig. 17.

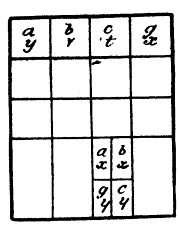

Fig. 18.

Fig. 17 shows a 4×4 square in the upper row of which the elements are written as previously given under Class I, Genus A. Filling the inner pair of cells in the lower row we see that these cells must contain $a + v$ and $c + y$, but we have the *choice* of writing $c + y$ in the right or left-hand cell. Choosing the right-hand cell the square is then completed by the laws previously given, and but slight attention is required to show that the contents of each cell is *forced* by these laws.

This square will be magic for any values assigned to the elements, and it will be normal if they are given the values 1, 2, 3, 4 and 0, 4, 8, 12.

To find the number of *possible* squares of the above class and genus we reflect that for the first cell we have a choice of 16. For the fourth cell we have a choice of only 4, since in the example, having used a and y in the first cell we are debarred from using either a, y, g or t in the fourth cell. Next, for the two central cells, we evidently have a choice of 4, and in completing the square we have the choice of *two* methods to fill the lower row. Multiplying the number of *choices* we have

$$16 \times 4 \times 4 \times 2 = 512.$$

and it is therefore clear that Class I, Genus A will yield 512 possible forms of squares.

Fig. 18 shows a square in which the basic row of elements are arranged so as to produce Class I Genus B. In filling the central cells of the lower row, it is found that the equivalent of $(a + y) + (g + x)$ must be used, and there are three such equivalents, viz.,

(1) $(a + x) + (g + y)$
(2) $(b + x) + (c + y)$
(3) $(b + y) + (c + x)$.

(3) however will be found impossible, leaving only (1) and (2) available. Choosing (1) it will be seen that there are two choices since $a + x$ may be located in either the right or left-hand of the two cells. Similarly if (2) is chosen, $b + x$ may be placed in either of these cells. Hence in, say, the right-hand central lower cell, there may be placed:

(1) $a + x$
(2) $g + y$
(3) $b + x$
(4) $c + y$

as shown in Fig. 18, and when one of these four pairs of elements is used the remainder of the square becomes fixed. It therefore follows that for the first cell of the basic row there is a choice of 16. For the fourth cell of same row there is a choice of 4. For the central cells of same row there is a choice of 4 and for the lower row there is a choice of 4. Multiplying these choices together we have

$$16 \times 4 \times 4 \times 4 = 1024.$$

which is the possible number of variations of Class I, Genus B.

Writing a basic row of Class I, Genus C as given in Fig. 16, we find that the equivalents of $(a + x) + (g + v)$ must be used to fill the central cells of the lower row. Because

$$a + g = b + c$$
$$\text{and } x + v = t + y$$

there are no less than sixteen pairs which may be made all equal to each other. Ten of these pairs however will be found unavailable, leaving six pairs to choose from, and since each of these six pairs may be located in either of the two cells, there is a choice of 12 different ways in which the lower row may be filled.

For the first cell of the basic row, there is naturally a choice of 16. For the fourth cell of the same row there is no choice, as this cell must be filled with the complements of the first cell. For the two middle cells of the basic row there is a choice of 4. Multiplying these choices together we have:

$$16 \times 4 \times 12 = 768,$$

which is the possible number of variations of Class I, Genus C.

a	*a*	*g*	*g*
x	*v*	*y*	*t*

Fig 19.

a	*g*	*a*	*g*
x	*v*	*y*	*t*

Fig. 20.

Proceeding now to Class II, a basic row may be formed as given in Fig. 19. It is evident that neither *a* nor *g* can be used in the lower row of the square, but as equivalents of

$$(a + v) + (g + y)$$

we may use either of the two couples:

$$(c + y) + (b + v)$$
$$(b + y) + (c + v),$$

and since either couple may be placed in either of two cells, there is a choice of 4 variations. To form the basic row, we have for the first cell a choice of 16 as before. For the fourth cell there is a choice of 6 seeing that one of the elements of the first cell must be located therein, coupled with any one of the three remaining elements of the opposite group. For the two inner cells there is a choice of 2. Hence for Class II we have:

$$16 \times 6 \times 2 \times 4 = 768 \text{ varieties.}$$

Class III has a basic row constituted as shown in Fig. 20. It will be found impossible to construct a magic square from the above basic row along the lines hitherto followed. Nevertheless, four varieties of squares may be constructed on every basic row of this class, on account of certain relations between the two groups of elements which have not as yet been considered. The squares may be made as herein before shown, but when completed they appear to be imperfect, as will be seen in Figs. 21, 22, 23, and 24 which

a x	g v	a y	g t
b t	a v	g x	c y
g y	c x	b v	a t
c v	b x	c t	b y

Fig. 21.

a x	g v	a y	g t
g y	a v	g x	a t
b t	c x	b v	c y
c v	b x	c t	b y

Fig. 22.

a x	g v	a y	g t
c v	b y	c t	b x
b t	c x	b v	c y
g y	a t	g x	a v

Fig. 23.

a x	g v	a y	g t
b t	b y	c t	c y
c v	c x	b v	b x
g y	a t	g x	a v

Fig. 24.

illustrate four squares built up on the foregoing basic row. These squares although seemingly imperfect, are not actually so on account of a peculiar relationship between the numbers 1, 2, 3, 4 and 0, 4, 8, 12.

Class III has for the first cell of the basic row a choice of 16, for the third cell a choice of 6, for the second cell a choice of 2, and of each of the above forms there are 4 variations. Hence we have:

a x	a v	g x	g v

Fig. 25.

1	13	4	16
14	12	5	3
8	2	15	9
11	10	7	6

Fig. 26.

1	13	4	16
8	12	5	9
14	2	15	3
11	10	7	6

Fig. 27.

$$16 \times 6 \times 2 \times 4 = 768 \text{ varieties}$$

in this class.

Class IV has a basic row as shown in Fig. 25, and the two middle cells of the lower row may be filled with either of the two couples

$$(b + t) + (c + y)$$
$$\text{or } (b + y) + (c + t)$$

thus permitting a choice of 4. Having a choice of 16 for the first cell of the basic row, and a choice of 4 for the two inner cells of this row, we have as a total:

$$16 \times 4 \times 4 = 256.$$

This square however has a peculiar property, owing to each couple of cells containing a pair of elements, which permits two variants to be made after each sub-basic row has been fixed. This property is illustrated in Figs. 26 and 27, in which both of the upper and lower rows are alike, and yet the squares are diverse. For class IV we therefore have:

$$256 \times 2 = 512 \text{ varieties.}$$

Summarizing the preceding results it will be seen that there are in

Class	I,	Genus	A,	512	varieties
"	I,	"	B,	1024	"
"	I,	"	C,	768	"
"	II,			768	"
"	II,			768	"
"	III,			768	"
"	IV,			512	"
			Total	4352	varieties.

There are thus at least 4352 diverse forms of 4×4 magic squares which may be constructed with the numbers 1 to 16.

Passing on to the 5×5 square, its analysis may be omitted, as the principles that underlie the formation of the 3×3 and 4×4 squares also enter largely into the formation of this and all other squares.

It may therefore be taken for granted that the components may be formed by the successive addition of five qualities in one group to the five qualities in another group.

In order that the 5×5 square may consist of the numbers 1 to 25 inclusive, the following values may be assigned to the respective symbols:

$a = 0$	$x = 1$
$b = 1$	$y = 6$
$c = 2$	$t = 11$
$d = 3$	$s = 16$
$g = 4$	$v = 21.$

Other values might be used but the foregoing are probably best adapted for general purposes.

A brief study of this square will indicate that the basic row may be constructed in a very large number of different ways.

For the first cell there is a choice of 25 combinations of elements, for the second cell 16, for the third 9, for the fourth, 4, while for the fifth there is naturally no choice, there being only one available combination left. We therefore have at the very least

$$25 \times 16 \times 9 \times 4 = 14400$$

possible variations in the basic row of this square.

To complete the square, there are at least three available plans, and the resulting squares may be designated as Classes I, II and III respectively.

Class I is made by writing the symbols of the "*a*" group of elements in diagonal columns across the square, in one direction, say from left to right, and the symbols of the "*x*" group also in diagonal columns, but in the opposite direction as shown in Fig. 28.

a x	b y	c t	d s	g v
b v	c x	d y	g t	a s
c s	d v	g x	a y	b t
d t	g s	a v	b x	c y
g y	a t	b s	c v	d x

Fig. 28.

b v				
		b	v	
	v			b
	b			v
		v	b	

Fig. 29.

a x				b v
	x	a	a'	
	a'	x		a
	a		x	a'
		a'	a	x

Fig. 30,

It will be seen that the "*a*" elements occupy the right-hand diagonal columns and the "*x*" elements the left-hand diagonal columns. It is also evident that irrespective of the way in which the basic row may be filled, the square may be completed by making "*a*" or "*x*" elements occupy either the right or left-hand diagonal column, and hence there is a choice of *two* methods; in the one case, the center cell being filled with $a + v$, and in the other (as shown in Fig. 28) it is filled with $g + x$.

Class II. The Squares in this class are constructed by making the elements in the basic row move by "knight's moves." For example, if the left-hand corner cell of the basic row contains the symbols of $b + v$, these symbols (and also all the other components of the basic row) may assume the relative positions shown in Fig. 29. It is clear that in this case there is the option of exchanging the places of $b + v$ throughout the square, thus giving a choice of *two* ways.

Class III is made by combining the method of Class I with that of Class II as shown in Fig. 30. One element (and naturally all of its fellows in the group) runs diagonally while the other element is placed by knight's moves. There is consequently a choice of *two* elements, either of which may dominate the diagonals or be located by knight's moves. In the case shown in Fig. 30 it is evident that "a" may occupy the cells marked "a" or those marked "a^1," there being eight possible knight's moves from any cell. This fact gives still another choice of two different ways, so there is a choice of *four* methods in Class III.

Summarizing the foregoing results:

For the basic row there is a choice of 14400.

For Class I squares	2×14400	=	28800
For Class II squares	2×14400	=	28800
For Class III squares	4×14400	=	57600
			115200.

So there are at least 115200 different ways in which a perfect 5×5 square may be made.

According to the figures herein given the number of variations of the different sizes of squares that have been considered increases as follows:

3×3	square	1
4×4	"	4,352
5×5	"	115,200.

NOTES ON NUMBER SERIES USED IN THE CONSTRUCTION OF MAGIC SQUARES.

It has long been known that magic squares may be constructed from a series of numbers which do not progress in arithmetical order. Experiment will show, however, that any haphazard series cannot be used for this purpose, but that a definite order of sequence is necessary which will entail certain relationships between different members of the series. It will therefore be our endeavor in the present article to determine these relationships and express the same in definite terms.

Let Fig. 31 represent a magic square of 4×4. By rule No. 4 in the "New Analysis of Magic Squares" it is seen that "*the sum of the terms in any two contiguous corner cells is equal to the sum of* *the terms in the two middle cells in the opposite outside column."* Therefore, in Fig. 31, $a + d = v + s$, and it therefore follows that $a - v = s - d$. In other words, these four quantities form a group with the interrelationship as shown. By the same rule (No. 4) it is also seen that $a + t = l + p$, and hence also, $a - l = p - t$, giving another group of four numbers having the same form of interrelationship, and since both groups have "*a*" as an initial number, it is evident that the increment used in one of these groups must be different from that used in the other, or duplicate numbers would result. It therefore follows that the numbers composing a magic square are not made up of a single group, but necessarily of more than one group.

Since we have seen that the term "*a*" forms a part of two groups, we may write both groups as shown in Fig. 32, one horizontally and the other perpendicularly.

Next, by rule No. 5, it is shown that *"the sum of the two end*

a	*b*	*c*	*d*
g	*h*	*k*	*l*
m	*n*	*o*	*p*
t	*v*	*s*	*y*

Fig. 31.

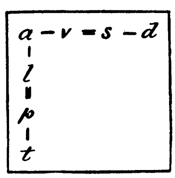

Fig. 32.

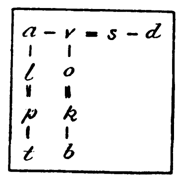

Fig. 33.

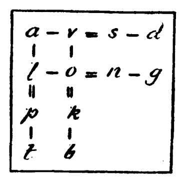

Fig. 34.

terms of any inside column (either horizontal or perpendicular) is equal to the sum of the two middle terms in the other parallel column." It therefore follows that $v + b = k + o$ or $v - o = k - b$. Using the term v as the initial number, we write this series perpendicularly as shown in Fig. 33. In the same way it is seen that $l + g = n + o$, or $l - o = n - g$, thus forming the second horizontal column in the square (Fig. 34). Next $p + m = h + k$ or $p - k = h - m$, forming the third horizontal column and in this simple manner the square may be completed as shown in Fig. 35.

It is therefore evident that a 4×4 magic square may be formed of any series of numbers whose interrelations are such as to permit them to be placed as shown in Fig. 35.

The numbers 1 to 16 may be so placed in a great variety of ways, but the fact must not be lost sight of that, as far as the construction of magic squares is concerned, they only *incidentally* possess the quality of being a single series in straight arithmetical order, being really composed of as many groups as there are cells in a column of the square. Unless this fact is remembered, a clear conception of the quantities of the series cannot be formed.

In illustration of the above remarks, three diagrams are given in Figs. 36, 37 and 38. Figs. 36 and 37 show arrangements of the numbers 1 to 16 from which the diverse squares Figs. 39 and 40 are formed by the usual method of construction.

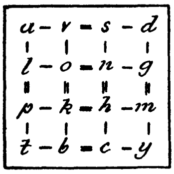

Fig. 35.

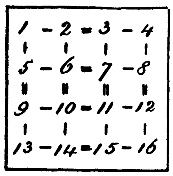

Fig. 36.

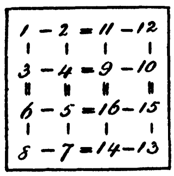

Fig. 37.

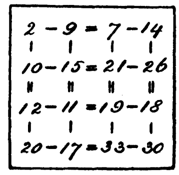

Fig. 38.

Fig. 38 shows the arrangement of an irregular series of sixteen numbers, which, when placed in the order of magnitude run as follows:

2-7-9-10-11-12-14-15-17-18-19-20-21-26-30-33

The magic square formed from this series is given in Fig. 41.

In the study of these number series the natural question presents itself: *Can as many diverse squares be formed from one series as from another?* This question opens up a wide and but little explored region as to the diverse constitution of magic squares. This

1	14	15	4
8	11	10	5
12	7	6	9
13	2	3	16

Fig. 39.

1	7	14	12
10	16	5	3
15	9	4	6
8	2	11	13

Fig. 40.

2	17	33	14
26	19	11	10
18	21	15	12
20	9	7	30

Fig. 41.

1	11	6	16
14	13	4	3
7	2	15	10
12	8	9	5

Fig. 42.

idea can therefore be merely touched upon in the present article, examples of several different plans of construction being given in illustration and the field left at present to other explorers.

Three examples will be given, the first being what is sometimes termed a "perfect" square, or one in which any two numbers that are geometrically opposite and equidistant from the center of the square will be equal in summation to any other pair of numbers so situated. The second example will be a square in which the sum of every diagonal of the four sub-squares of 2×2 is equal, and the third example will be a square in which the pairs of numbers having similar summations are arranged symmetrically in relation to a perpendicular line through the center of the square. Figs. 39, 40 and 42 illustrate these three examples of squares.

Returning now to the question previously given, but little reflection is required to show that it must be answered in the negative for the following reasons. Fig. 41 represents a magic square having

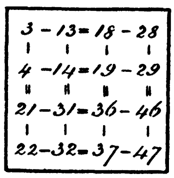

Fig. 41.

3	32	37	28
29	36	31	4
46	19	14	21
22	13	18	47

Fig. 42.

no special qualities excepting that the columns, horizontal, perpendicular and diagonal all have the same summation, viz., 66. Hence *any* series of numbers that can be arranged as shown in Fig. 35 will yield magic squares as outlined. But that it shall also produce squares having the qualifications that are termed "perfect," may or may not be the case accordingly as the series may or may not be capable of still further arrangement.

1	4	7	10	13
8	11	14	17	20
15	18	21	24	27
22	25	28	31	34
29	32	35	38	41

Fig. 43.

25	38	1	14	27
35	13	11	24	22
10	8	21	34	32
20	18	31	29	7
15	28	41	4	17

Fig. 44.

Referring to Fig. 31, if we amend our definition by now calling it a "perfect" square, we shall at once introduce the following continuous equation:

$$a+y=h+o=t+d=n+k=b+s=c+v=g+p=m+l,$$

and if we make our diagram of magic square producing numbers conform to these new requirements, the number of groups will at once be greatly curtailed.

The multiplicity of algebraical signs necessary in our amended diagram is so great that it can only be studied in detail, the complete diagram being a network of minus and equality signs.

The result will therefore only be given here, formulated in the following laws which apply in large measure to all "perfect" squares.

I. Perfect magic squares are made of as many series or groups of numbers as there are cells in a column.

II. Each series or group is composed of as many numbers as there are groups.

III. The differences between any two adjoining numbers of a series must obtain between the corresponding numbers of all the series.

IV. The initial terms of the series compose another series, as do the second, third, fourth terms and so on.

V. The differences between any adjoining numbers of these secondary series must also obtain between the corresponding terms of all the secondary series.

The foregoing rules may be illustrated by the series and perfect square shown in Figs. 36 and 39.

Following and consequent upon the foregoing interrelations of these numbers is the remarkable quality possessed by the "perfect" magic square producing series as follows:

If the entire series is written out in the order of magnitude and the differences between the adjacent numbers are written below, the row of differences will be found to be geometrically arranged on each side of the center as will be seen in the following series taken from Fig. 43.

3 - 4 -13-14-18-19-21-22-28-29-31-32-36-37-46-47
1 9 1 4 1 2 1 (6) 1 2 1 4 1 9 1

In the above example the number 6 occupies the center and the other numbers are arranged in geometrical order on each side of it. It is the belief of the writer that this rule applies to all "perfect" squares whether odd or even.

The following example will suffice to illustrate the rule as applied to a 5×5 magic square, Fig. 45 showing the series and Fig. 46 the square.

1 . 4 . 7 . 8 . 10. 11. 13. 14. 15. 17. 18. 20. 21. 22. 24. 25. 27. 28. 29. 31. 32. 34. 35. 38. 41
3 3 1 2 1 2 1 1 2 1 2 1 | 1 2 1 2 1 1 2 1 2 1 3 3

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