A GEOMETRIC REPRESENTATION.*

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§ 1. Introduction.

In order to visualize the complex values of y, when such exist, of a plane curve $y=f(x)$, or a surface $y=f(x, z)$, and also for the purpose of representing some real curves in space by a single independent equation in x and y, I adjoin an ordinary complex plane, perpendicular to the x axis of the real xy plane, with its real axis parallel to the y axis, in fact always in the real plane and with its origin in the axis of x, so that the complex plane slides along always perpendicular to the x axis, OX, and at distance x from O the origin of the xy plane, as x changes. By this representation the equation of every curve or surface has an actual and uninterrupted locus from $-\infty$ to $+\infty$, including the usual real locus of $y=f(x)$ or $y=f(x, z)$, and some real curves in space can be represented by a single independent equation between two variables x and y. I prefer this method to that of projection used by Phillips and Beebe and others, because that method shows both the real curve and the complex portion in the same (real) plane and intersecting in points in which they do not intersect. Moreover, projection on a plane would be of no use in the case of surfaces. Projection on a fixed plane perpendicular to the x axis in case the complex curves lie on a cylinder whose axis is parallel to OX would not give a one-to-one correspondence and hence is not always available. The reason why we do not ordinarily get the complete locus, that is, the usual real and complex portion taken together, is that we have arbitrarily shut out the complex portion by the (usual real) kind of representation that we have chosen to use. While forms of representation for the complex portions of curves are known, it is believed that the method here used of representing surfaces and of visualizing the portions of their complex loci is new, as well as the representation of certain real

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curves in space by a single independent equation between x and y.

It is easily seen that the complete locus of the equation $x^2 + y^2 = a^2$ is, in addition to the circle, an equilateral hyperbola $x^2 - y^2 = a^2$ touching the circle at $(\pm a, 0)$ but turned out of the real plane through an angle of 90° about OX. The imaginary asymptotes $x^2 + y^2 = 0$, of the circle, are the actual asymptotes of the equilateral hyperbola and lie in its plane and pass through the real point (0, 0), of the real plane. Similarly $x^2 - y^2 = a^2$ has for its complex locus the circle $x^2 + y^2 = a^2$ turned through 90°. In fact $x^2 + y^2 = a^2$ and $x^2 - y^2 = a^2$ are complementary to each other when one is turned through an angle of 90°. In a similar way

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are complementary. Complementary to $y^2 = 2mx$ is a congruent parabola turned through 90° about OX and 180° about OY, and touching the former at (0, 0). The cissoid

$$y^2 = \frac{x^3}{2a - x} \quad \text{and} \quad y^2 = \frac{x^3}{x - 2a}$$

are similarly complementary in the same way, cusp touching cusp point on at a common tangent. Likewise the lemniscate $(x^2+y^2)^2 = a^2(x^2-y^2)$ and the curve $(x^2-y^2)^2 = a^2(x^2+y^2)$ are complementaries when one is turned through 90° about OX. They touch at $(\pm a, 0)$.

The complex values of y as function of x have in general a different geometric representation from the complex values of x as function of y. This means only that the impossibility of being in the real plane exists in different ways along the two axes. In the case of some functions there is only one representation, no impossibility existing in the other direction. In the case of others the representation is of the same kind in both directions; y is the same function of x that x is of y. We consider the representation of y as function of x in this paper. If we consider complex representations in both directions "complete locus" will be accordingly enlarged. In the above illustrations we considered only the complete locus of y as function of x. For surfaces analogous statements hold. In this paper it is shown how to obtain the equation of any surface in a form adapted to this representation, the equation of a family of spirals on a surface, the equation of the surface on which a family of spirals lies, and the length of an arc on a spiral, with illustrations. Next the application of the representation to the investigation of the function $y=(x^{x-1}(x-1)^{x})^{1/(1-2x)}$, brought to my attention by Professor F. W. Very, and for which a graph was required, forms the major part of the paper. The special properties, the value systems with their sequence and contiguity, and the spiral systems of the function are considered in detail.

§ 2. The Equation of a Surface Adapted to this Representation.

In rectangular co-ordinates the equation of the surface is $f(x, y, z)=0$. y and z are the real and purely imaginary parts of the complex y represented in § 1. The equation of the surface in this representation is $y=|y|e^{\phi i}$.

$$|y|=(y^2+z^2)^{\frac{1}{2}}=((F_1(x, \phi))^2+(F_2(x, \phi))^2)^{\frac{1}{2}}$$

where

$$y=F_1(x, \phi), \quad z=F_2(x, \phi),$$

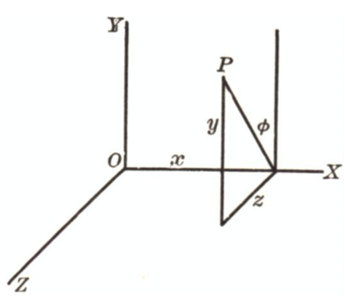

Fig. 1.

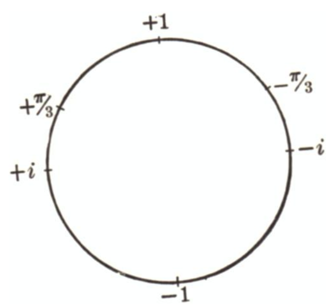

Fig. 2.

ϕ is the angle from the plane XOY to the plane XOP, Fig. 1; it is the amplitude or argument of the complex y of our representation. ϕ and x are independent variables. $z=y\tan\phi$, and $y=z\operatorname{ctn}\phi$. $F_1(x, \phi)$ is found by solving $f(x, y, y\tan\phi)=0$ for y, $F_2(x, \phi)$ by solving $f(x, z\operatorname{ctn}\phi, z)=0$ for z. The case where these solutions are algebraically or practically impossible is left for consideration in a future paper. The equation of the surface is then

$$y=((F_1(x, \phi))^2+(F_2(x, \phi))^2)^{\frac{1}{2}}e^{\phi i} \qquad (1)$$

and it is seen that the locus exists from $x=-\infty$ to $x=\infty$.

Examples.

1. The equation of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

is

$$\begin{aligned} y &= bc\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}\left(\frac{1}{c^2+b^2\tan^2\phi}+\frac{1}{c^2\operatorname{ctn}^2\phi+b^2}\right)^{\frac{1}{2}}e^{\phi i}, \\ &= R(x,\phi)e^{\phi i}. \end{aligned} \qquad (2)$$

2. The equation of the bi-parted hyperboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

is

$$\begin{aligned} y &= bc\left(\frac{x^2}{a^2}-1\right)^{\frac{1}{2}}\left(\frac{1}{c^2+b^2\tan^2\phi}+\frac{1}{c^2\operatorname{ctn}^2\phi+b^2}\right)^{\frac{1}{2}}e^{\phi i}, \\ &= S(x, \phi)e^{\phi i}. \end{aligned} \qquad (3)$$

3. From examples 1 and 2 we note $S=iR$, $R=-iS$. From these examples follow easily the representation of the complete ellipsoid and the complete bi-parted hyperboloid. For the ellipsoid $y=R(x, \phi)e^{\phi i}$ when $|x|>|a|$ the equation becomes $y=iS(x, \phi)e^{\phi i}=S(x, \phi)e^{(\phi+\frac{\pi}{2})i}$, and for the bi-parted hyperboloid $y=S(x, \phi)e^{\phi i}$—when $|x|<|a|$, the equation takes the form

$$y=iR(x, \phi)e^{\phi i}=R(x, \phi)e^{(\phi+\frac{\pi}{2})i}.$$

Since $y=S(x, \phi)e^{(\phi+\frac{\pi}{2})i}$ is the bi-parted hyperboloid (where each y has an amplitude $\phi+\frac{\pi}{2}$ instead of ϕ) turned through 90° about OX, and in the same manner $y=R(x, \phi)e^{(\phi+\pi)i}$ is the ellipsoid turned through 90°, it is shown that the ellipsoid has for its complex portion a bi-parted hyperboloid in a space turned through 90° about OX, and that the bi-parted hyperboloid has an ellipsoid turned through 90° for its complex part. It is not to be supposed that the interpretations for the complex portions of all surfaces will be as simple as these or that all the ordinates together will be turned through 90°.

§ 3. Spirals on the Surface $f(x, y, z)=0$.

We have seen in § 2 that the equation of the surface $f(x, y, z)=0$ is in our representation

$$y=((F_1(x, \phi))^2+(F_2(x, \phi))^2)^{\frac{1}{2}}e^{\phi i}.$$

If now in the place of ϕ we substitute a function of x, $f(x)$, we shall evidently have the equation of a line on the surface expressed by the single independent equation in x and y,

$$y=((F_1(x, f(x)))^2+(F_2(x, f(x)))^2)^{\frac{1}{2}}e^{f(x)i}, \qquad (4)$$

for as x increases, $f(x)$ either (1) increases, (2) decreases, or (3) is constant.

(1) If x increases, the amplitude $f(x)$ increases, the point $P=(x, y)$ continually winds around the surface tracing a right-handed spiral when viewed from O looking towards OX.

(2) If $f(x)$ decreases, the spiral is left-handed.

(3) If $f(x)$ is constant, the spiral degenerates into the intersection of the surface by a plane embracing OX.

If $f(x)$ increases for a while and then decreases, the spiral is right-handed for a while and then reverses and goes on as a left-handed spiral. If $f(x)=\sin kx$, the spiral can be made to be a fluctuating wabbly curve on the surface varying in shape according to the value taken for k.

Examples.

1. A spiral on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

has the equation

$$y=R(x, f(x))e^{f(x)i}, \qquad (5)$$

and when x reaches the values $\pm a$, the spiral does not stop but goes on to the complementary bi-parted hyperboloid and goes to infinity on its surface.

2. Similarly the spiral

$$y = S(x, f(x))e^{f(x)i} \tag{6}$$

on the bi-parted hyperboloid goes on the surface of the complementary ellipsoid between $x = -a$ and $x = a$.

3. If $a = \infty$ in example 1, § 2,

$$y = bc\left(\frac{1}{c^2 + b^2 \tan^2 \phi} + \frac{1}{c^2 \operatorname{ctn}^2 \phi + b^2}\right)^{\frac{1}{2}} e^{\phi i}, \tag{7}$$

the equation of an elliptic cylinder whose axis is OX, and whose semi-axes are b and c. If $b = c = a$,

$$y = ae^{\phi i}, \tag{8}$$

the equation of a circular cylinder, as could have been written down directly without this derivation. The equation of a spiral on the elliptic cylinder whose axis is OX and whose semi-axes are a and b is

$$y = ab\left(\frac{1}{a^2 + b^2 \tan^2 f(x)} + \frac{1}{a^2 \operatorname{ctn}^2 f(x) + b^2}\right)^{\frac{1}{2}} e^{f(x)i} \tag{9}$$

and the equation of a spiral on a circular cylinder is

$$y = ae^{f(x)i} \tag{10}$$

and as a special case of the last the equation of the helix is

$$y = ae^{kxi}. \tag{11}$$

(To be continued.)